PhD Proposal

Information and the Dispersion of Posterior Distributions

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1 Introduction

Uncertainty exists nearly everywhere in the economy, like the uncertain price of a stock, the fluctuating exchange rate, or the unobservable skill of a worker. The recognition of imperfect information has had a profound influence and has provided a remarkable method for explaining economic and social phenomena. Although they may not be sure about the true state, agents can, in most cases, overcome the uncertainty by obtaining related information that is conveyed in some signals, which may be derived from personal investigation, or suggested by an expert, or purchased from some institution, or even stolen. Their decision will then be based on the revised knowledge about the true state, i.e. the posterior belief of the state. As depicted in Figure 1, agents make decisions before knowing the true state but after observing an informative signal which is correlated to the true state.

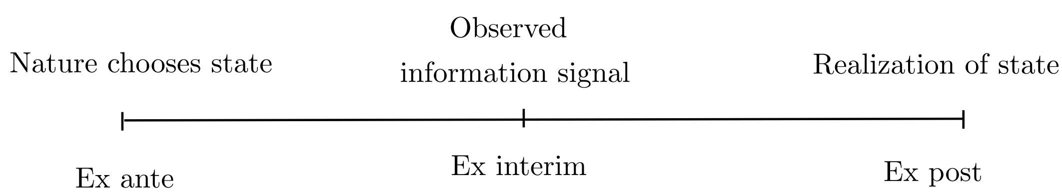

Figure 1: Timing of events

But how could we tell the quality of the information? How would distinctive signals influence the decision process? Consider an extreme case where the signals could accurately reveal the future state, then the revised beliefs will depend completely on the realized value of the signal rather than the prior beliefs. On the other hand, if the informational content of the signal is relatively low, the posterior belief will be similar as the prior belief. That is to say, the posterior distribution is supposed to be less disperse under less informative signals. This could also be extended to the conditional expectation of one's payoff. So more information should lead to a more disperse conditional expectations. When the signal is fully uninformative , there would be no revision of belief - posterior beliefs is the same as prior beliefs, which yields that conditional expectations equal to the unconditional one for any realization of the signal - no dispersion at all. An simple illustration is shown in Figure 2.

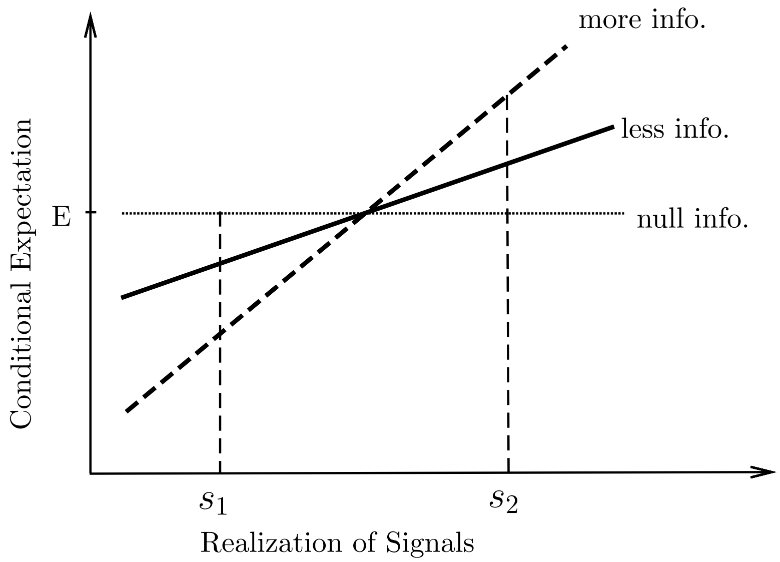

Figure 2: Conditional expectations derived from different informative signals

Therefore, to investigate the relation between the informativeness of signal and the dispersion of posterior distribution, as well as the dispersion of conditional expectations becomes a focus of this study.

2 Statement of the Problem

2.1 Information Structures

In the setting of statistical decision theory, uncertainty can be characterized by a random state of nature. As stated before, a decision maker takes actions before knowing the true state but after observing a signal which is correlated to the true state. We further assume that the agent has a prior probability distribution of the state of nature, then he can infer additional information from the signal and revise his belief about the true state via Bayes'

Rule. In this context, an information structure is used to formalize how the signals are generated by the true state.

Definition 1 *The triplet* (Ω, Y, F) *is defined as an* ***information structure****, where* Ω *is the set of states,* Y *is the set of signals, and* F *is a stochastic transformation from* Ω *to* Y*, represented by the conditional density functions* $f(y|\omega)$*, for any* $y \in Y$ *and* $\omega \in \Omega$*.*

In fact, for a certain decision problem, an information structure mainly refers to the family of probability density functions (or distributions) of the signals conditional on the state of nature. Then, for a given prior $\pi(\cdot)$ defined on Ω, agents can update their beliefs; the posterior belief is given by:

$$\nu(\omega|y) = \frac{f(y|\omega)\pi(\omega)}{\mu(y)}, \quad \forall \omega \in \Omega, \forall y \in Y.$$

where $\mu(y) = \int_\Omega f(y|\omega')\pi(\omega')d\omega'$. Decisions then will be made according to the posterior distributions.

The availability of additional information allows agents to better react to the risky environment, which leads naturally to the consideration of the value of information, or of a certain information structure. Consider an expected utility maximizer with a utility function $u(a, \omega)$, where $a \in A$ is the action the agent takes. After receiving a signal y, the agent chooses the optimal action $a^*(y)$ which solves the maximization problem $max_{a \in A} \int_{\omega \in \Omega} u(a, \omega)\nu(\omega|y)d\omega$. And the value of the information is

$$V := \int_{y \in Y} \mu(y) \int_{\omega \in \Omega} u(a^*(y), \omega)\nu(\omega|y)d\omega dy$$

That is, the value of information is the agent's ex ante expected utility from an optimal chosen decision rule. From this it is also feasible to compare two information structures with different informative signals.

2.2 The Ordering of Information structures

Various criteria have been proposed during the past few decades aiming at evaluating two information structures in terms of the informativeness of the generated signals. One classical criterion was developed by Blackwell (1953) which follows the intuition that a less informative information structure can be duplicated from a more informative one by adding some random transmission error. Marschak and Miyasawa (1968) use the term "garbling" for this stochastic signal transformation process. Formally,

Definition 2 *Let (Ω, Y^F, F) and (Ω, Y^G, G) be two information structures. (Ω, Y^F, F) is said to be* ***more informative*** *than (Ω, Y^G, G), denoted by $(\Omega, Y^F, F) \succ^i (\Omega, Y^G, G)$, if there exists a stochastic transformation Γ from Y^F to Y^G represented by a stochastic density kernel γ such that for all $\omega \in \Omega$ and $y^H \in Y^H$, it holds true that*

$$g(y^G|\omega) = \int_{y^F \in Y^F} \gamma(y^G, y^F) f(y^F|\omega) dy^F.$$

Although the order based on Blackwell's criterion is only a partial order, the ranking is independent of particular prior beliefs and the assumption of utility functions. Indeed, the more informative structure is preferred by every expected utility maximizer, i.e. $V^F \geq V^G$, as stated in the well-known Blackwell's Theorem.

Large literature follows the Blackwell's informativeness criterion[1], yet alternative criteria have also been introduced. For example, Lehmann (1988) considers information structures with monotone likelihood ratio property (MLRP)[2], which indicates that higher signals are more favourable[3]. He proposed the so-called effectiveness criterion from the intuition that a better information should be more correlated with the true state. More specifically, provided the signals are related to one another, the structure F yields more information than G, if for a given signal from G, the higher the state is, the higher the signal under F is. Besides, Lehmann also demonstrates that Persico (2000) formalizes Lehmann's criterion as an accuracy criterion, followed also by Jewitt (2007) who relates

[1] See Le Cam (1964), Ponssard (1975), Crémer (1982), Schlee (2001), Eckwert and Zilcha (2003), etc.

[2] The densities $\{f(y|\omega)\}_{y \in Y, \omega \in \Omega}$ have the monotone likelihood ratio property, if for every $y' > y$ and $\omega' > \omega$, $f(y'|\omega')f(y|\omega) - f(y'|\omega)f(y|\omega') \geq 0$.

[3] According to Milgrom (1981), a signal y' is more favourable than another signal y if for every non-degenerate prior distribution $\pi(\cdot)$ for ω, the posterior distribution $\nu(\cdot|y')$ dominates the posterior distribution $\nu(\cdot|y)$ in the sense of strict first-order stochastic dominance.

the Blackwell's criterion with Lehmann's and extends Lehmann's results in a general single crossing preference (SCP) case. More recently, Quah and Strulovici (2009) even generalize the results to the cases where SCP does not hold.

In addition, Kim (1995) defines a efficiency criterion that higher informativeness is in correspondence with a mean preserving spread (MPS)[4] of the likelihood ratio distribution function. He also shows that Blackwell's criterion implies the efficiency criterion.

2.3 Dispersion Criteria

Following the intuition that signals containing more useful information have a stronger impact on posterior distributions, which may in turn yield more disperse expectations conditional on the signals, Ganuza and Penalva (2010) propose a new kind of criteria for evaluating different information structures, which they refer to as precision criteria. Two precision criteria - the supermodular precision and the integral precision - are focused in their study, which are specified on the basis of different stochastic orders. For instance, the supermodular precision based on the dispersive order is defined as: A signal $\tilde{y}^F$ generated from (Ω, Y^F, F) is more *supermodular precise* than another signal $\tilde{y}^G$ from (Ω, Y^G, G), if $E[\tilde{\omega}|\tilde{y}^F]$ is greater in the dispersive order [5] than $E[\tilde{\omega}|\tilde{y}^G]$.

In fact, the dispersive order requires necessarily that the more disperse random variable has a broader support. Thus, in order to compare any realizations of different signals, Ganuza and Penalva (2010) show an alternative characterization of supermodular precision by introducing a transformed signal which is defined as $\tilde{z}^k = F^k(\tilde{y}^k)$, $k \in \{F, G\}$[6], where F^k is the cumulative distribution function of $\tilde{y}^k$. According to the probability integral transform [7], the new signal is uniformly distributed on the interval (0,1), regardless

[4]Rothschild and Stiglitz (1970) apply MPS as a tool of risk measurement, which provides a new perspective to order information structures.

[5]Y and Z are two real-valued random variables with distributions F and G, respectively. Then, Y is said to be greater than Z in the dispersive order if for all $q, p \in (0, 1)$, $q > p$, we have $F^{-1}(q) - F^{-1}(p) \geq G^{-1}(q) - G^{-1}(p)$.

[6]Similar transformation of signals can also be found in Eckwert and Zilcha (2008).

[7]The probability integral transform theorem states that if X is a random variable with continuous

the original distribution of signal $\tilde{y}$[8]. Therefore, note that $E^k[\tilde{\omega}|z] = E[\tilde{\omega}|\tilde{z}^k = z]$, the supermodular precision criterion can also be characterized as below.

Definition 3 *$\tilde{y}^F$ is more* ***supermodular precise*** *than $\tilde{y}^G$ if*

$$E^F[\tilde{\omega}|z'] - E^F[\tilde{\omega}|z] \geq E^G[\tilde{\omega}|z'] - E^G[\tilde{\omega}|z]$$

for any $z, z' \in (0, 1)$ such that $z' > z$.

Definition 3 describes the relation between signal precision and the conditional expectations in a more explicit way. It is obvious that the difference between the two conditional expectations, denoted by $\Delta E(z) := E^F[\tilde{\omega}|z] - E^G[\tilde{\omega}|z]$, is monotonically increasing in z. Moreover, the conditional expectation for the more precise information structure $E^F[\tilde{\omega}|z]$ is always steeper than that obtained in the less precise one, $E^G[\tilde{\omega}|z]$, which means that the conditional expectation is more sensitive to the changes in signal realizations if the signal is more precise.

Similar as the supermodular precision criterion, the integral precision is also defined on conditional expectations but with another stochastic order - the convex order. Therefore, some other orders may also be considered to define the precision of information structure, such as mean-preserving spread, single-crossing dispersion, rotation, etc.

2.4 Purpose of the Study

Although the formalization of precision criteria provides an easier way to interpret the informativeness of signals, it does not possess the property that the ordering is invariant to the relabelling of the unknown states, and unconventionally, the ordering by the precision criteria is based on conditional expectations rather than directly on the underlining information structures. So could it be possible to link the precision criteria with

cumulative distribution function $F_X(x)$, then $U = F_X(X)$ is uniformly distributed over the interval (0,1).

[8] The transformed signal is uniformly distributed only when the distribution function F is continuous and strictly increasing, yet Lehmann (1988) provides a construction of an equivalent signal, which can solve the problem of the discontinuity of the distribution function.

the traditional informativeness criteria? Is there some implications among these criteria? Therefore, to bridge the gap between the informativeness and the dispersion criteria is one of the purposes of this study.

A preliminary examination on binary information structures shows that the Blackwell's criterion does imply a dispersion of conditional expectations, but it does not hold for general cases without any other restrictions. Thus, further characterizations of the dispersion criteria would also be important if one would like to apply the criteria.

Moreover, the establishment of the link between informativeness and dispersion criteria would form a theoretical basis to analyse different kinds of economic decision problems by applying directly the dispersion criteria, such as the investment on education, employment decision in the labour market, or portfolio investment in the financial market, etc.

3 Methodology and Objectives

The investigation of the issues stated above is mainly related to the subjects of probability theory and mathematical statistics, comparative statics analysis and the dynamic equilibrium theory. With these theoretical tools, study on the ranking criteria of information structures will be conducted, followed by the application on examining how the precision of the information affects the equilibrium of financial market. Different types of investment projects and investors with a variety of investment behaviours will be considered in the model. To be specific, the following issues will be included in this research.

- Establish the relation between informativeness and dispersion criteria.
- Form Characterizations of dispersion criteria.
- Applications on financial market based on dispersion criteria (market transparency and financing probabilities, expected returns, welfare and efficiency analysis, policy interventions, market (in)stability, etc.)

4 Proposed Time-Table

Mar. 2012	Familiarization with different stochastic orders and informativeness criteria; Preliminary investigation on discrete cases.
Aug. 2012	Formalization of the relation between informativeness and dispersion criteria.
Dec. 2012	Characterizations of characterizing the dispersion criteria.
Dec. 2013	Analysis of investment behaviour on financial markets.
Jul. 2014	Elaboration and completion of the dissertation.

References

[1] Blackwell, D. Equivalent comparisons of experiments. *Annals of Mathematical Statistics* 24 (1953), 265-272.

[2] Crémer, J. A simple proof of Blackwell's "comparison of experiments" theorem. *Journal of Economic Theory* 27 (1982), 439-443.

[3] Eckwert, B., and Zilcha, I. Incomplete risk sharing arrangements and the value of information. *Economic Theory* 21 (2003), 43-58.

[4] Eckwert, B., and Zilcha, I. Efficiency of screening and labour income inequality. *Journal of Public Economic Theory* 10, 1 (2008), 77-98.

[5] Ganuza, J., and Penalva, J. S. Signal ordering based on dispersion and the supply of private information in auctions. *Econometrica* 78, 3 (2010), 1007-1030.

[6] Jewitt, I. Information order in decision and agency problems. Personal manuscript, Nuffield College, University of Oxford.

[7] Kim, S. K. Efficiency of an Information System in an Agency Model. *Econometrica* 63, 1 (1995), 89-102.

[8] Le Cam, L. Sufficiency and approximate sufficiency. *Annals of Mathematical Statistics* 35 (1964), 1419-1455.

[9] Lehmann, B. L. Comparing location experiments. *Annals of Statistics* 16, 2 (1988), 521-533.

[10] Marschak, J and Miyasawa, K. Economic Comparability of Information Systems. *International Economic Review* 9, 2 (1968), 137-174.

[11] Milgrom, P. R. Good news and bad news: Representation theorems and applications. *Bell Journal of Economics* 12 (1981), 380-391.

[12] Persico, N. Information Acquisition in Auctions. *Econometrica* 68, 1 (2000), 135-148.

[13] Ponssard, J.-P. A Note on Information Value Theory for Experiments Defined in Extensive Form. *Management Science* 22, 4 (1975), 449-454.

[14] Quah, J. K.-H. and Strulovici, B. Comparative Statics, Informativeness, and the Interval Dominance Order. *Econometrica* 77, 6 (2009), 1949-1992.

[15] Rothschild, M and Stiglitz, J. E. Increasing risk: I. A definition. *Journal of Economic Theory* 2 (1970), 225-243.

[16] Schlee, E. E. The Value of Information in Efficient Risk-Sharing Arrangements. *American Economic Review* 91, 3 (2001), 509-524.